Number Inequalities

1. Which number is greater:
    a. 2^{300} or 3^{200}?
    b. 2^{40} or 3^{28}?
    c. 5^{45} or 4^{54}?
    d. 31^{11} or 17^{14}?
    e. $\frac{1998}{1999}$ or $\frac{1999}{2000}$?
    f. $\sqrt{19}+\sqrt{99}$ or $\sqrt{20}+\sqrt{98}$?
2. Prove that $2^{100}+3^{100}<4^{100}$.
3. Prove that $\frac{1}{2}<\frac{1}{101}+\frac{1}{102}+...+\frac{1}{200}<1$.
4. Prove that $\frac{1}{2}<1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{199}-\frac{1}{200}<1$.
5. Prove that $\frac{1}{101}+\frac{1}{102}+...+\frac{1}{200}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{199}-\frac{1}{200}$.
6. Prove that $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{500,000}<20$.
7. Find a value of n such that $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}>20$.

Triangle Inequalities

For any triangle *ABC* with sides *a*, *b* and *c*:

i. $a < b + c, \quad b < a + c, \quad c < a + b$

ii. $a > |b - c|, \; b > |a - c|, \; c > |a - b|$

iii. $(a + b - c)(b + c - a)(c + a - b) > 0$

1. In how many ways can we choose three different numbers from the set $\{1,2,3,4,5,6\}$ such that the three could be the sides of a triangle? (Note: the order of the chose numbers doesn't matter.)
2. Side *AC* of triangle *ABC* has length 3.8, and side *AB* has length 0.6. If the length of side *BC* is an integer, what is its length?
3. Find all positive integers *x* for which it is possible for $2x + 3$, $3x + 8$ and $6x + 7$ to be the side lengths of a triangle.
4. Prove that the length of any side of a triangle is not more than half its perimeter.
5. Let *AM* be a median of triangle *ABC*. Prove that $AM > \dfrac{AB + AC - BC}{2}$.
6. Find a point inside a convex quadrilateral such that the sum of the distances from the point to the vertices is minimal.
7. Prove that the sum of the diagonals of a quadrilateral is less than the quadrilateral's perimeter.
8. Prove that the sum of the diagonals of a quadrilateral is greater than half the perimeter of the quadrilateral.
9. Point *O* lies outside square *ABCD*. Prove that the distance from *O* to one of the vertices of the square is not greater than the sum of the distances from *O* to the other three vertices.
10. A fly sits on one vertex of a wooden cube. What is the shortest path it can follow to the opposite vertex?
11. A fly sits on the outside surface of a cylindrical drinking glass. It must crawl to another point, situated on the inside surface of the glass. Find the shortest path possible (neglecting the thickness of the glass).
12. A hunter leaves the woods at point *A* in the figure. He must reach the road, which follows a straight line, and go back into the woods at point *B*. If he wants to travel the shortest distance, how should he do this?
13. A woodsman's hut is in the interior of a peninsula that has the form of an acute angle. The woodsman must leave his hut, walk to one shore of the peninsula, then to the other shore, and then return home. How should he choose the shortest such path?
14. Point *C* lies inside a given right angle, and points *A* and *B* lie on its sides (see figure). Prove that the perimeter of triangle *ABC* is not less than twice the distance *OC*, where *O* is the vertex of the given right angle.
15. Two villages lie on opposite sides of a river whose banks are parallel lines. A bridge is to be built over the river, perpendicular to the banks. Where should the bridge be built so that the path from one village to the other is as short as possible?

Arithmetic and Geometric Means

1. Prove that the arithmetic mean of two numbers, $\frac{a+b}{2}$, corresponds to the point on the number line that lies halfway between *a* and *b*.
2. The arithmetic mean of two numbers 1 and *a* is equal to 7. Find *a*.
3. The geometric mean of two numbers 1 and *a* is equal to 7. Find *a*.
4. Find the side of a square having the same perimeter as a rectangle with sides *a* and *b*. Find the side of a square having the same area as a rectangle with sides *a* and *b*.
5. Prove that for nonnegative *a* and *b* $\sqrt{ab} \le \frac{a+b}{2}$.
6. When is he arithmetic mean of two numbers equal to their geometric mean?
7. What is the maximum value of the product of two nonnegative numbers whose sum is a fixed positive number *c*? What is its minimum value?
8. What are the maximum and minimum values of the sum of two nonnegative numbers whose product is a fixed $c > 0$?
9. What is the maximum possible area of a rectangular piece of land if you may enclose it with only 120m of fence?
10. What is the maximum possible area of a rectangular piece of land bordering the (straight) seashore if you may enclose it with only 120m of fence?
11. What is the maximum value of the product *ab* if *a* and *b* are nonnegative numbers such that $a + 2b = 3$?